The Journal of the Astronautical Sciences, Vol. 35, No. 1, January-March 1987, pp. 3–18

The Behavior of Long Tethers in Space[1]

David A. Arnold[2]

Abstract

The behavior of satellites connected by long tethers is discussed in terms of basic physical principles with some mathematics included. The topics included the gravitational, centrifugal, and aerodynamic forces on the system, vertical stabilization using the gravity gradient force, librations of the system, and longitudinal and transverse motions of the tether. Deployment and retrieval of the system are discussed, along with strategies for controlling librations particularly during retrieval. Other topics include the use of tethers for exchanging energy and momentum between satellites, tether strength requirements and tapering techniques in long or rotating systems, and instability of a particular type of extremely long tethered configuration.

Introduction

The purpose of this paper is to discuss the dynamics of long orbiting tethers in terms of basic principles of physics and mathematics. Many of the most important aspects of the behavior of tethers can be understood in a simple way without resorting to complicated mathematics. Computer simulations using only basic principles can be written to study the dynamics using numerical integration. Deriving the equations of motion in a reference frame rotating with the orbit is more complicated but produces useful insight into the characteristics of the motion.

Vertical Gravity Gradient

A satellite orbiting the Earth is subjected to a gravitational force F_g given by

$$F_g = -\frac{GMm}{r^2} \tag{1}$$

where GM is the gravitational constant of the Earth (3.986013×10^{20} in cgs units or 3.986013×10^{14} in MKS units), m is the mass of the satellite and r is the distance from the center of the Earth. The centrifugal force F_c is given by

[1]Presented at the NASA/AIAA/PSN International Conference on Tethers in Space, Arlington, VA, September 1986.

[2]Mr. David A. Arnold is affiliated with the Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, MA 02138.

$$F_c = mr\Omega^2 \tag{2}$$

where Ω is the orbital angular velocity. In a circular orbit, the net vertical force F_z on the mass given by

$$F_z = F_g + F_c \tag{3}$$

must be zero. Setting $F_z = 0$ gives the relation

$$\Omega^2 = \frac{GM}{r^3} \tag{4}$$

Figure 1 shows a tethered system orbiting the Earth such that the tether remains aligned with the local vertical and all parts of the system have the same orbital angular velocity Ω. The lower mass m_1 will be subjected to more gravitational force and less centrifugal force than the upper mass m_2. The result is that there is a net force toward the Earth on m_1 and a net force away from the Earth on m_2. This results in a tension T in the wire. In order for the system to remain in a circular orbit, the angular velocity Ω must be such that the total gravitational force on the system is equal and opposite to the total centrifugal force on the system. Therefore we must have

$$\int_{r_1}^{r_2} \frac{GMdm}{r^2} = \int_{r_1}^{r_2} dmr\Omega^2 \tag{5}$$

Solving for Ω^2 gives

$$\Omega^2 = \int_{r_1}^{r_2} \frac{GMdm}{r^2} \bigg/ \int_{r_1}^{r_2} dmr \tag{6}$$

If we can neglect the mass of the tether, equation (5) becomes simply

$$\frac{GMm_1}{r_1^2} + \frac{GMm_2}{r_2^2} = m_1 r_1 \Omega^2 + m_2 r_2 \Omega^2 \tag{7}$$

from which Ω^2 can be easily calculated. There will be some point along the system where the gravitational and centrifugal forces are equal to each other. A mass at this point is in a zero-g condition and experiences no net force in the radial direction. The value of the orbital radius r_o where this condition occurs can be calculated by taking the value of Ω^2 from equation (6) and putting it into equation (4), which gives the

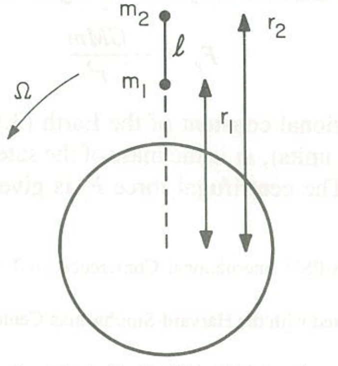

FIG. 1. Vertically Stabilized Tether System.

relationship between Ω and r required to have equal gravitational and centrifugal forces. Setting $r = r_o$ we have

$$\Omega^2 = \frac{GM}{r_o^3} \tag{8}$$

or

$$r_o = \left(\frac{GM}{\Omega^2}\right)^{1/3} \tag{9}$$

For short tethers the value of r_o will be nearly equal to the position of the center of mass of the system. However the difference between r_o and the position of the center of mass can be significant for long tethered systems.

The force F_z on a mass m located on a tethered system at radius r from the center of the earth can be calculated using equation (3) with F_g given by equation (1), F_c by equation (2) and Ω^2 by equation (6). At r_o the value of F_z is zero. We can derive an approximate expression for F_z as a function of the distance from r_o in the following way. The rate of change of F_g with respect to r obtained by differentiating equation (1) is

$$\frac{\partial F_g}{\partial r} = 2\frac{GMm}{r^3} \tag{10}$$

The rate of change of F_c is, from equation (2)

$$\frac{\partial F_c}{\partial r} = m\Omega^2 \tag{11}$$

If $r = r_o$, equation (8) can be used to rewrite equation (10) as

$$\frac{\partial F_g}{\partial r} = 2m\Omega^2 \tag{12}$$

If a mass is at a distance z from r_o, where z is given by

$$z = r - r_o \tag{13}$$

the force F can be written approximately as

$$F_z = z\left(\frac{\partial F_g}{\partial r} + \frac{\partial F_c}{\partial r}\right) \tag{14}$$

Substituting equations (11) and (12) gives

$$F_z = 3m\Omega^2 z \tag{15}$$

Equation (15) gives an approximate formula for the gravity gradient force on a mass at distance z from the zero-g point r_o along the tether. From equations (11) and (12) we see that the "gravity gradient" force F_z actually consists of two parts gravity gradient, and one part centrifugal gradient.

Out-of-Plane Gravity Gradient Force

Suppose two masses are in orbit at the same altitude, but separated from each other in the out-of-plane direction as shown in Fig. 2. We wish to calculate the gravity gradient force acting along the line between the two masses. Figure 3 shows the system as seen by an observer looking in the direction of motion. The behavior of masses separated in the out-of-plane direction can be understood by considering the orbits of the masses as free satellites. The velocities of m_1 and m_2 in Fig. 3 are parallel and directed into the plane of the paper. Since the gravity force is directed toward the center of the Earth the plane of the orbit of m_1 is different from that of m_2. Starting from the positions shown in Fig. 3, the masses would meet after 1/4 of an orbit. In a free orbit, the centrifugal force $\mathbf{F}_c$ on m_1 is equal and opposite to the gravitational force $\mathbf{F}_g$. Suppose masses m_1 and m_2 are held together by a rigid structure. They would then be forced to move in parallel circles and the centrifugal force $\mathbf{F}_c'$ on m_1 would be perpendicular to the y axis as shown in Fig. 3. The gravitational force would then have an unbalanced component in the y direction. It is more convenient to use F_c given by equation (2) to compute the net force since it is written as a function of Ω. The net force F_y on mass 2 is

$$F_y = F_g \frac{y}{r} = -F_c \frac{y}{r} = -mr\Omega^2 \frac{y}{r} \tag{16}$$

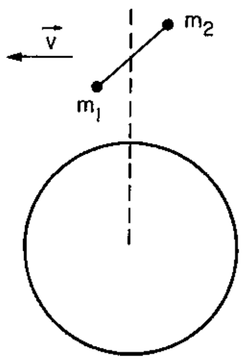

FIG. 2. Two Masses at the Same Altitude with Out-of-Plane Displacement.

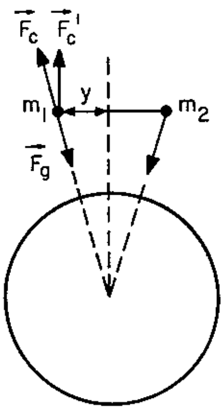

FIG. 3. Two Out-of-Plane Masses Seen from the Direction of Motion.

$$F_y = -m\Omega^2 y \tag{17}$$

The gravity gradient force in the out-of-plane direction is negative and does not produce a tension in a tether connecting the masses.

In-Plane Gravity Gradient Force (Flight Direction)

There is no net in-plane force. Figure 4 shows two masses separated from each other in the in-plane direction at the same orbital altitude. Since both masses have the same orbital plane, there is no relative motion between them. They simply follow each other around in the orbit maintaining a constant relative distance. The centrifugal and gravitational forces remain equal and opposite. (A rigorous derivation of the equations of motion in a rotating frame shows a centrifugal acceleration component $\Omega^2 x$ in the in-plane direction that is cancelled by the in-plane component $-\Omega^2 x$ of the gravity gradient acceleration.) We can write the gravity gradient force in the in-plane direction as

$$F_x = 0 \tag{18}$$

Librations

From the preceding sections, we can summarize the gravity gradient forces acting on a tethered satellite as follows. At the orbital center of a system, the gravitational and centrifugal forces are equal and opposite. If a mass is moved and held fixed at some point along the x axis (flight direction) F_g and F_c remain equal and opposite. If the displacement is in the z direction (along the local vertical) the forces remain opposite but not equal. In the y direction (out-of-plane) the forces remain equal but not opposite. Let us define a coordinate system with origin at r_o (the zero-g point of the system), having the z-axis pointing away from the Earth, the x-axis pointing in the direction of the orbital velocity, and y-axis pointing in the out-of-plane direction as shown in Fig. 5. From equations (15), (17) and (18) we have the vector gravity gradient force on a mass given by the set of equations

$$F_x = 0 \tag{19}$$

$$F_y = -m\Omega^2 y \tag{20}$$

$$F_z = 3m\Omega^2 z \tag{21}$$

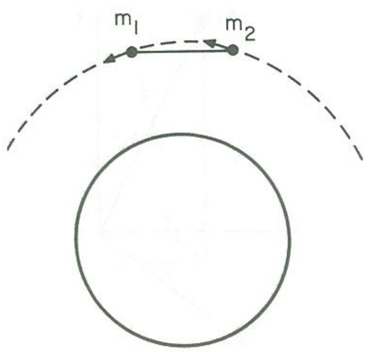

FIG. 4. In-Plane Displacement (Flight Direction).

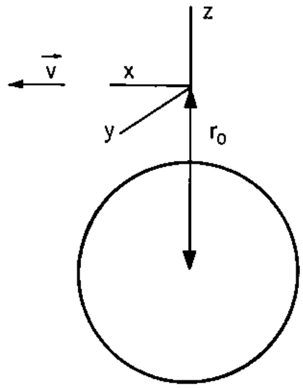

FIG. 5. Rotating Coordinate System.

Suppose we have a tether system of length ℓ that is deployed upward with small displacements from the vertical in the x and y directions as shown in Fig. 6. There will be a gravity gradient force of magnitude $3m\Omega^2\ell$ in the z direction and a force $-m\Omega^2 y$ in the y direction. Let us define an in-plane angle θ and an out-of-plane angle ϕ where θ and ϕ are given by the equations

$$\theta \approx x/\ell \tag{22}$$

and

$$\phi \approx y/\ell \tag{23}$$

for small displacements. The gravity gradient force F_z will produce torques affecting the in-plane and out-of-plane angles, and the gravity gradient force F_y will produce an additional torque on the out-of-plane angle. The torque τ_θ on the in-plane angle is

$$\tau_\theta = -\ell F_z \theta = -3m\Omega^2\ell^2\theta \tag{24}$$

The torque τ_ϕ for the out-of-plane angle is

$$\tau_\phi = -\ell F_z \phi + \ell F_y = -3m\Omega^2\ell^2\phi - m\Omega^2\ell y \tag{25}$$

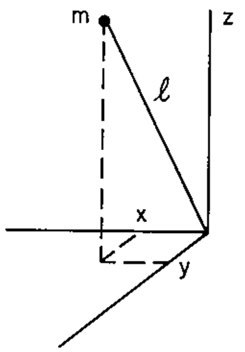

FIG. 6. Libration of the Tether.

Since $y = \phi \ell$ we can write

$$\tau_\phi = -4m\Omega^2\ell^2\phi \tag{26}$$

We see that there is a restoring torque in both the in-plane and out-of-plane directions that tends to keep the system aligned with the local vertical. The restoring torque for the out-of-plane angle is stronger than that of the in-plane angle. As a result, out-of-plane librations have a higher frequency than in-plane librations. Equations of motion for the system have been derived and the results are presented in the Appendix. If the angular deviations and angular velocities are small, the only important terms in the equations of motion for θ and ϕ are the torques given by equations (24) and (26). Since $\tau_\theta = m\ell^2\ddot{\theta}$, and $\tau_\phi = m\ell^2\ddot{\phi}$, the equations of motion for small angles and fixed tether length are

$$\ddot{\theta} = -3\Omega^2\theta \tag{27}$$

and

$$\ddot{\phi} = -4\Omega^2\phi \tag{28}$$

The frequency of the in-plane libration is $\sqrt{3}\ \Omega$ and the frequency of the out-of-plane libration is 2Ω, where Ω is the orbital angular rate.

Momentum Exchange Using Tethers

Figure 7 shows a vertically stabilized tethered system. Mass m_1 is at an orbital radius r_1 and mass m_2 is at an orbital radius r_2. The zero-g point of the system is at orbital radius r_o. All parts of the system move with constant orbital angular velocity Ω which can be calculated from equation (6). For short systems, r_o is approximately the position of the center of mass, and one can calculate Ω approximately by setting r_o equal to the

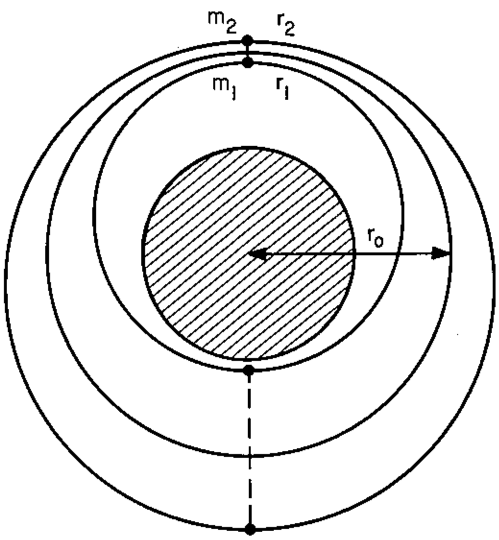

FIG. 7. Orbits of Masses After Release from a Tether.

orbital radius of the center of mass in equation (8). The orbital speeds v_1 and v_2 of masses m_1 and m_2 are

$$v_1 = \Omega r_1 \tag{29}$$

and

$$v_2 = \Omega r_2 \tag{30}$$

The gravitational and centrifugal forces are equal at orbital radius r_o. The gravitational force on m_1 is stronger and the centrifugal force is weaker. If m_1 were released from the end of the tether, it would drop into a lower orbit with the point of release being the apogee of the new orbit. For mass m_2 the gravitational force is weaker and the centrifugal force is stronger. If it were released from the end of the tether it would go into a higher orbit with the point of release being the perigee of the new orbit.

After release from the end of the tether a mass will go into an elliptical orbit. The semi-major axis a of the new orbit can be calculated from the equation

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{31}$$

where v is the velocity, r is the distance from the center of the earth and GM is the gravitational constant of the earth. The velocity at the time of release is given by equations (29) and (30). For short tether lengths, the change in orbital altitude after half an orbit is given approximately by the equation

$$(r' - r_o) \approx 7(r - r_o) \tag{32}$$

where r is the distance from the center of the earth at the release point, and r' is the distance after half an orbit. If the mass is below the orbital center r_o, r' is the perigee of the new orbit. If the mass is released from above the orbital center, r' is the new apogee. Equation (32) was derived using equation (31). The details of the derivation are given in Reference [1].

Velocity Dependent Forces

Equations (19) through (21) give the forces on a mass as a function of the displacement from the orbital center when there is no motion relative to the orbital center. If the mass has a velocity relative to the rotating orbital coordinate system there can be additional forces. Figure 5 defines a rotating orbital coordinate system with the x-axis in the direction of the orbital motion. Suppose a mass has a velocity $\dot{x}$ with respect to the rotating coordinate system. The orbital centrifugal force is given by equation (2). At the orbital center this is equal and opposite to the gravitational force given by equation (1). However the velocity $\dot{x}$ is added to the orbital velocity **v** shown in Fig. 5, thereby increasing the orbital angular velocity Ω. This results in an additional centrifugal force in the z direction. The change in the orbital centrifugal force can be obtained by differentiating equation (2) with respect to Ω which gives

$$dF_c = 2mr\Omega d\Omega \tag{33}$$

The change in Ω is

$$d\Omega = \frac{\dot{x}}{r_o} \tag{34}$$

Setting $r = r_o$ in equation (33) and substituting equation (34) for $d\Omega$ gives

$$dF_c = 2m\Omega\dot{x} \tag{35}$$

If there is a velocity $\dot{z}$ in the radial direction there will be a Coriolis force F_{COR} in the $-x$ direction given by

$$F_{COR} = -2m\Omega\dot{z} \tag{36}$$

A velocity $\dot{y}$ in the out-of-plane direction does not produce either a Coriolis force or an additional centrifugal force. These results can be summarized by the equations

$$F'_x = -2m\Omega\dot{z} \tag{37}$$

$$F'_y = 0 \tag{38}$$

$$F'_z = 2m\Omega\dot{x} \tag{39}$$

where F'_x, F'_y and F'_z are the velocity dependent forces due to motion relative to the rotating orbital coordinate system.

Combining equations (19) through (21) and (37) through (39), the equation of motion in the rotating orbital coordinate system is

$$\begin{aligned} \mathbf{F}_e = {} & m\hat{\mathbf{x}}(\ddot{x} + 2\Omega\dot{z}) \\ & + m\hat{\mathbf{y}}(\ddot{y} + \Omega^2 y) \\ & + m\hat{\mathbf{z}}(\ddot{z} - 2\Omega\dot{x} - 3\Omega^2 z) \end{aligned} \tag{40}$$

where $\mathbf{F}_e$ is any external applied force. The equations of motion given in the Appendix contain additional centrifugal, Coriolis and coupling terms resulting from the use of a spherical coordinate system.

Deployment and Retrieval

Equations (19) through (21) and (37) through (39) give sufficient information to discuss the technique of retrieval or deployment at a constant in-plane angle. Suppose we have a vertically stabilized orbiting tether system such as shown in Fig. 1, which we wish to retrieve. Let us assume for this discussion that m_2 is very heavy compared to m_1 so that the orbital center is close to the position of m_2. The principles discussed would apply equally well to the case of comparable masses by using the orbital center as the origin of a rotating coordinate system. Suppose mass m_2 starts to pull in on the tether so that mass m_1 is given a velocity $\dot{z}$ in the vertical direction. From equation (37) we see that there will be a Coriolis force pushing m_1 toward the $-x$ direction. The mass m_1 will start to acquire an in-plane libration angle θ as shown in Fig. 8. Using equation (21) we can calculate the gravity gradient force in the z direction. If the tether length is ℓ, the z position is

$$z = -\ell \cos \theta \tag{41}$$

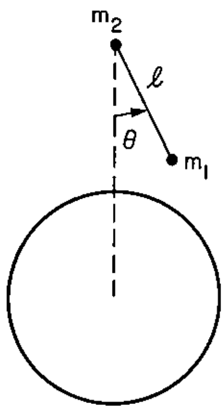

FIG. 8. Retrieval at a Constant In-Plane Angle.

so that the gradient force is

$$F_z = -3m\Omega^2\ell \cos\theta \qquad (42)$$

This will have a component F_θ in the $\hat{\boldsymbol{\theta}}$ direction given by

$$F_\theta = F_z \sin\theta = -3m\Omega^2\ell \cos\theta \sin\theta \qquad (43)$$

Figure 9 shows the components of the retrieval velocity $\dot{\ell}$ in the x and z directions. The velocity components are

$$\dot{x} = -\dot{\ell} \sin\theta \qquad (44)$$

and

$$\dot{z} = -\dot{\ell} \cos\theta \qquad (45)$$

Substituting equations (44) and (45) into equations (37) through (39) gives the velocity dependent forces

$$F'_x = 2m\Omega\dot{\ell} \cos\theta \qquad (46)$$

and

$$F'_z = -2m\Omega\dot{\ell} \sin\theta \qquad (47)$$

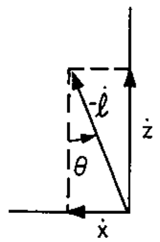

FIG. 9. Retrieval Velocity Components.

The component F'_θ of these forces in the $\hat{\boldsymbol{\theta}}$ direction is

$$F'_\theta = -2m\Omega\dot{\ell}\cos^2\theta - 2m\Omega\dot{\ell}\sin^2\theta = -2m\Omega\dot{\ell} \tag{48}$$

Retrieval at a constant angle θ can be achieved by choosing $\dot{\ell}$ such that the velocity dependent forces F'_θ are equal and opposite to the gradient forces F_θ. Setting equations (43) and (48) equal and opposite gives

$$2m\Omega\dot{\ell} = -3m\Omega^2\ell\cos\theta\sin\theta \tag{49}$$

or

$$\dot{\ell} = -\frac{3}{2}\ell\Omega\cos\theta\sin\theta \tag{50}$$

Equation (50) can also be obtained from the general equation of motion in the Appendix by setting $F_\theta = \ddot{\theta} = \dot{\theta} = \phi = \dot{\phi} = 0$. The equation is of the form

$$\frac{\dot{\ell}}{\ell} = -\alpha \tag{51}$$

where

$$\alpha = \frac{3}{2}\Omega\cos\theta\sin\theta \tag{52}$$

Equation (51) is easily integrated to give

$$\ell = \ell_0 e^{-\alpha t} \tag{53}$$

where ℓ_0 is the length at $t = 0$. Deployment at a constant angle can be achieved by having the deployment velocity proportional to ℓ.

Control Strategies During Retrieval

Deployment of a subsatellite is an inherently stable operation. Retrieval is inherently unstable, and cannot be accomplished without special control techniques. The application of a rate control law such as equation (50) will not produce a stable retrieval because small oscillations will tend to grow in amplitude as the length of the tether decreases. Because of the Coriolis forces given by equations (37) through (39), there is strong coupling between the rate of change of tether length and the in-plane angle. Increasing the retrieval rate will increase the in-plane libration angle, and decreasing the rate will decrease the angle. It is therefore possible to control the in-plane angle by adjusting the retrieval rate. There is poor coupling between the retrieval rate and the out-of-plane libration angle. For the out-of-plane librations, reasonable retrieval times can be achieved by active control of the out-of-plane angle using thrusters on either of the end masses. As the tether length becomes very short, an in-line thruster can be used to prevent loss of tension in the tether and active control of the in-plane angle can be substituted for rate control to avoid exponentially decreasing retrieval rates for the final approach. At some point atmospheric drag forces would become larger than the decreasing stabilization effect of the gravity gradient force. The final approach must be done under nearly zero-g conditions.

Wire Oscillations and Rotations of the Subsatellite

In addition to in-plane and out-of-plane librations which have been discussed previously, a tethered system can execute a number of other kinds of oscillation. The wire can have both longitudinal and transverse oscillations. If the mass of the tether is small compared to that of the end masses, the motion can be described by the usual equations for a vibrating string. In very long tethers, the mass of the tether can be substantial and the dynamics is more complicated. The end bodies can also rotate back and forth under the restoring torque of the wire. All of these motions can be coupled to each other in varying degrees. At low altitudes atmospheric drag can excite transverse wire oscillations and librations of the system. In an inclined orbit drag forces can produce a resonant buildup of out-of-plane librations. Analysis of the motions of the system requires rather complex computer simulation programs.

Strength of Tethers

Because of the length of the tethers to be used in space, it is important to minimize the total mass of the tether by using materials such as Kevlar which have a high strength to weight ratio. The tether cannot be made too thin because of the risk of being cut by a micrometeorite impact. A long tether can have a large total surface area so that the risk of a micrometeorite impact can be significant. The tethers planned for use on the TSS project have a diameter of about 2 mm and are made of a braided construction which should be able to survive most impacts so as to have a reasonable expected lifetime.

If the diameter of a tether is kept constant, there is a maximum attainable length in near Earth orbit beyond which the tether will break under its own weight because of the gravity gradient forces on the system. The greatest stress occurs at the orbital center of the system. The gravity gradient forces are directed away from the orbital center both in the upward and downward directions. At any point along the tether, the tether must be strong enough to support all the parts of the system in the direction away from the orbital center. Tethers of indefinite length can be put in orbit by tapering the tether so as to maintain a constant stress per unit cross sectional area. The difference in tension between the ends of a segment of tether is equal to the net force on the segment due to gravitational and centrifugal forces. The cross section of the tether should be proportional to the tension at each point. Since the gravity gradient is non-linear for long tethers, exact expressions should be used for the gravitational and centrifugal forces. If A is the cross section, ρ the density, and c the strength per unit cross section of the tether, the change in cross section dA over a segment of wire of length dr can be calculated from the equation

$$cdA = \frac{GM}{r^2}dm - r\Omega^2 dm \tag{54}$$

where $dm = \rho A dr$, r is the distance from the center of the Earth, Ω is the orbital angular velocity, and GM is the gravitational constant of the Earth. Below the orbital center A increases with r because the gravitational force is greater than the centrifugal force. Above the orbital center A decreases with r. The tether must be thickest at the orbital center and tapered toward the ends (see Fig. 10). Although there is no limit to

FIG. 10. Tapering of a Long Vertical Tether.

the length that can be achieved, the payload that can be supported on the end keeps getting smaller. Additional equations for studying such a system are given in Reference [1].

Rotating tethers are another example where tapering can save considerable tether mass. If a rotating tether is retrieved by pulling in the tether at one end, the centrifugal forces on the system will keep increasing. The last part of the tether retrieved will be subjected to the greatest stress and must be the thickest (see Fig. 11).

Altitude Changes During Deployment and Retrieval

In the process of retrieval of a tethered satellite system, the reel motor does work against gravity gradient forces. This work increases the orbital energy of the system. Conversely, during deployment, the orbital energy decreases. The gravity gradient forces are proportional to the tether length for short tethers, so that the work done during retrieval is proportional to the square of the tether length for short tethers. For long tethers the tension is no longer linear with tether length. Assuming that no thrusters or other external forces are used during retrieval, there must be conservation of the total angular momentum of the system. The final orbit after retrieval can be calculated using the principle of conservation of angular momentum. For simplicity one can assume that the retrieval is done slowly and that the final orbit is circular. The work done during retrieval can then be calculated as the difference in total orbital energy before and after retrieval. Figure 12 shows a plot of retrieval energy vs. tether length calculated in this manner. Both masses are 10 metric tons and the lower mass is at an altitude of 200 km before retrieval. The tether mass is neglected. Initially the retrieval energy is proportional to the square of the tether length. At a tether length of about 5000 km the

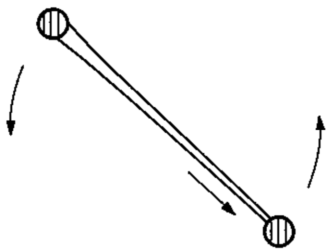

FIG. 11. Tapering of a Rotating Tether.

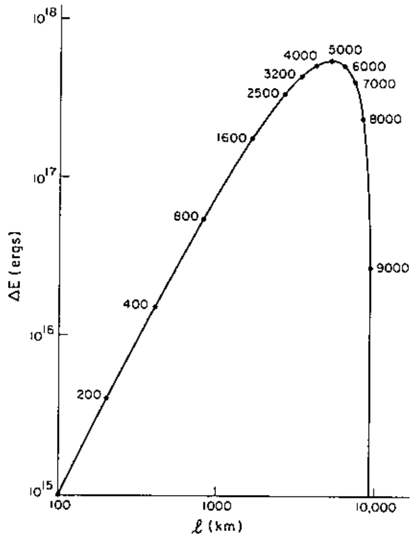

FIG. 12. Retrieval Energy vs. Tether Length.

retrieval energy reaches a maximum and decreases for longer tether lengths. Past about 9000 km the retrieval energy is negative. In other words, the total orbital energy after retrieval is less than the energy before retrieval. In trying to understand how this could happen, it was noticed that the total orbital energy before retrieval is positive for very long tethers. The system has enough energy to escape from orbit but is trapped in a circular orbit.

Computer simulations have been done to study the dynamics of these very long systems. The simulations show that these systems are unstable. As long as the system remains perfectly aligned with the local vertical it can remain in a circular orbit. Eventually, because of small errors in the initial conditions, vertical stabilization is lost. If the system librates forward (so that the libration angular velocity is parallel to the orbital angular velocity) a system with positive orbital energy will escape from orbit. If the system librates backward, the system falls out of orbit and the lower mass impacts the body about which the system is orbiting.

Table 1 lists various transition points as a function of the ratio of the orbital radius of the upper mass to that of the lower mass.

TABLE 1. Orbital Radius Ratios For Transition Points

r_2/r_1	
2.890053638	Total system energy equal to zero
2.385996517	Retrieval energy equal to zero
1.83929	Center of energy at the altitude of the upper mass
1.7556982	Maximum retrieval energy
1.521379707	Energy of upper mass equal to zero
1.451368226	Altitude after retrieval equal to that of the upper mass before retrieval

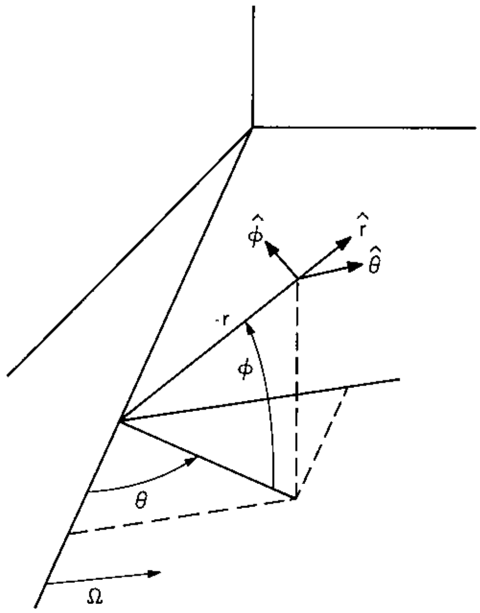

FIG. 13. Rotating Spherical Coordinate System.

Computer simulations have been done to study negative energy states to see if they are unstable for long tether lengths. For the idealized case used in the analysis, the simulations show that there is a gradual onset of instability starting at tether lengths in the vicinity of 6000 km. It is difficult to determine the exact length at which instability occurs. The center of energy goes outside the system at a tether length of 5520.83 km. This may provide an explanation for the onset of instability. These systems between about 5500 and 12,432 km have negative orbital energy so that they cannot escape. However they do not remain in a stable circular orbit. The details of the analysis and simulations are given in Reference [2].

Appendix

Figure 13 shows the position of a tethered object with respect to the orbital center of a system. It is assumed that the orbital center moves in a perfect circular orbit at constant angular velocity Ω. The general equations of motion for the system are

$$\begin{aligned}\mathbf{F} = {} & m\hat{\mathbf{r}}[\ddot{r} - r\dot{\phi}^2 - r\cos^2\phi(\dot{\theta} + \Omega)^2 + r\Omega^2 - 3r\Omega^2\cos^2\theta\cos^2\phi] \\ & + m\hat{\boldsymbol{\theta}}[\ddot{\theta}r\cos\phi + 2(\dot{\theta} + \Omega)(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) \\ & \qquad + 3r\Omega^2\cos\theta\cos\phi\sin\theta] \\ & + m\hat{\boldsymbol{\phi}}[r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\cos\phi\sin\phi(\dot{\theta} + \Omega)^2 + 3r\Omega^2\cos^2\theta\cos\phi\sin\phi]\end{aligned} \tag{A1}$$

where θ is the in-plane angle, ϕ is the out-of-plane angle, r is the distance from the orbital center, $\mathbf{F}$ is any applied force on the body, and m is the mass of the body. These equations are valid for short tether lengths since they were derived by linearizing the gravity force. The derivation of these equations, including exact expressions for the gravity force, is given in Reference [3].

References

[1] COLOMBO, G. and ARNOLD, D. A. "Study of Certain Launching techniques Using Long Orbiting Tethers," NASA Grant NAG-8008, March 1981.

[2] LORENZINI, E., ARNOLD, D. A. and GULLAHORN, G. E. "Study of an Orbiting Tethered Dumbbell System Having Positive Orbital Energy," NASA Contract NAS8-35497, February 1985.

[3] COLOMBO, G., ARNOLD, D. A. and DOBROWONLY, M. "Investigation of Electrodynamic Stabilization and Control of Long Orbiting Tethers," NASA Contract NAS8-33691.