Relativistic Positioning as a complementary technique of LASER Ranging

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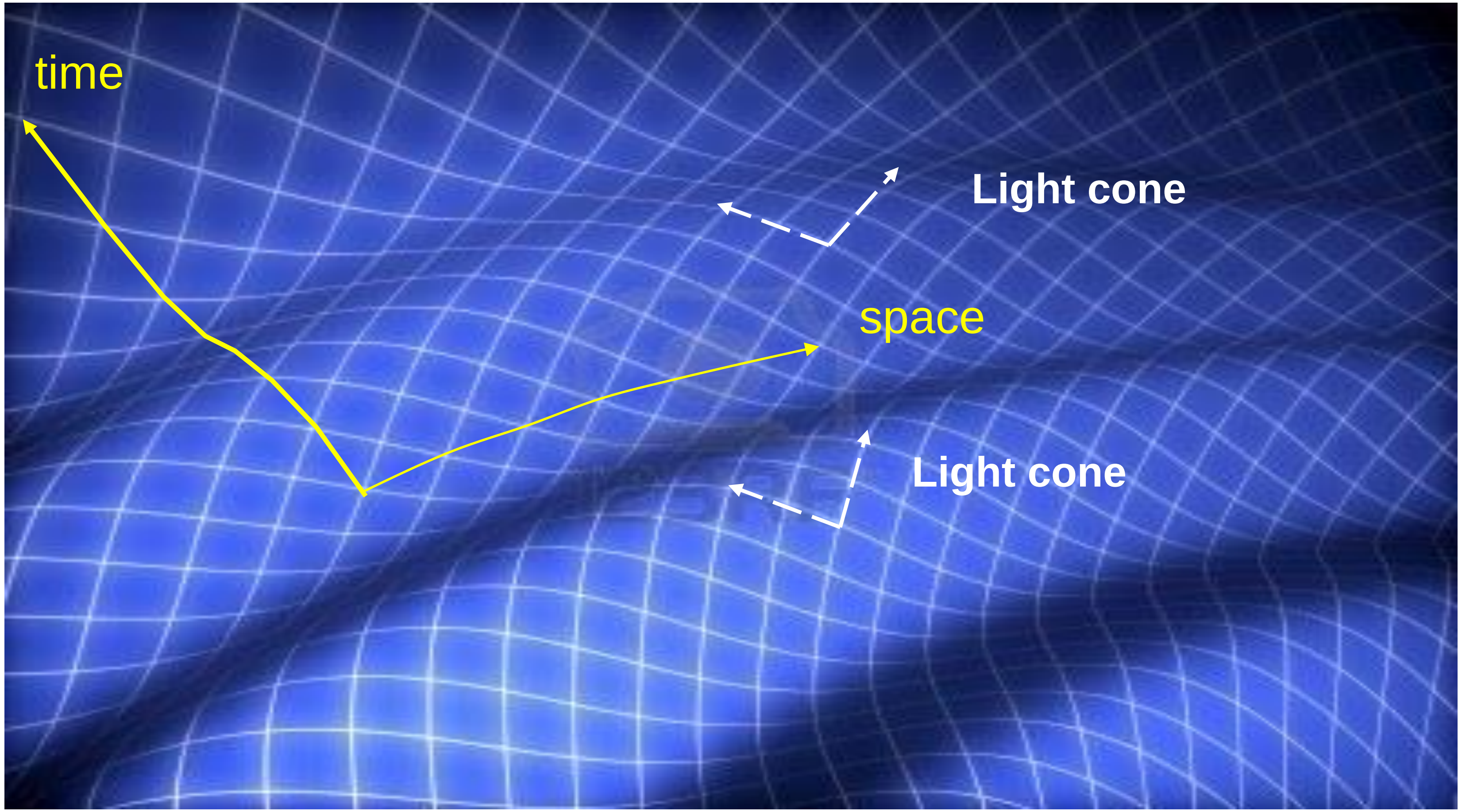
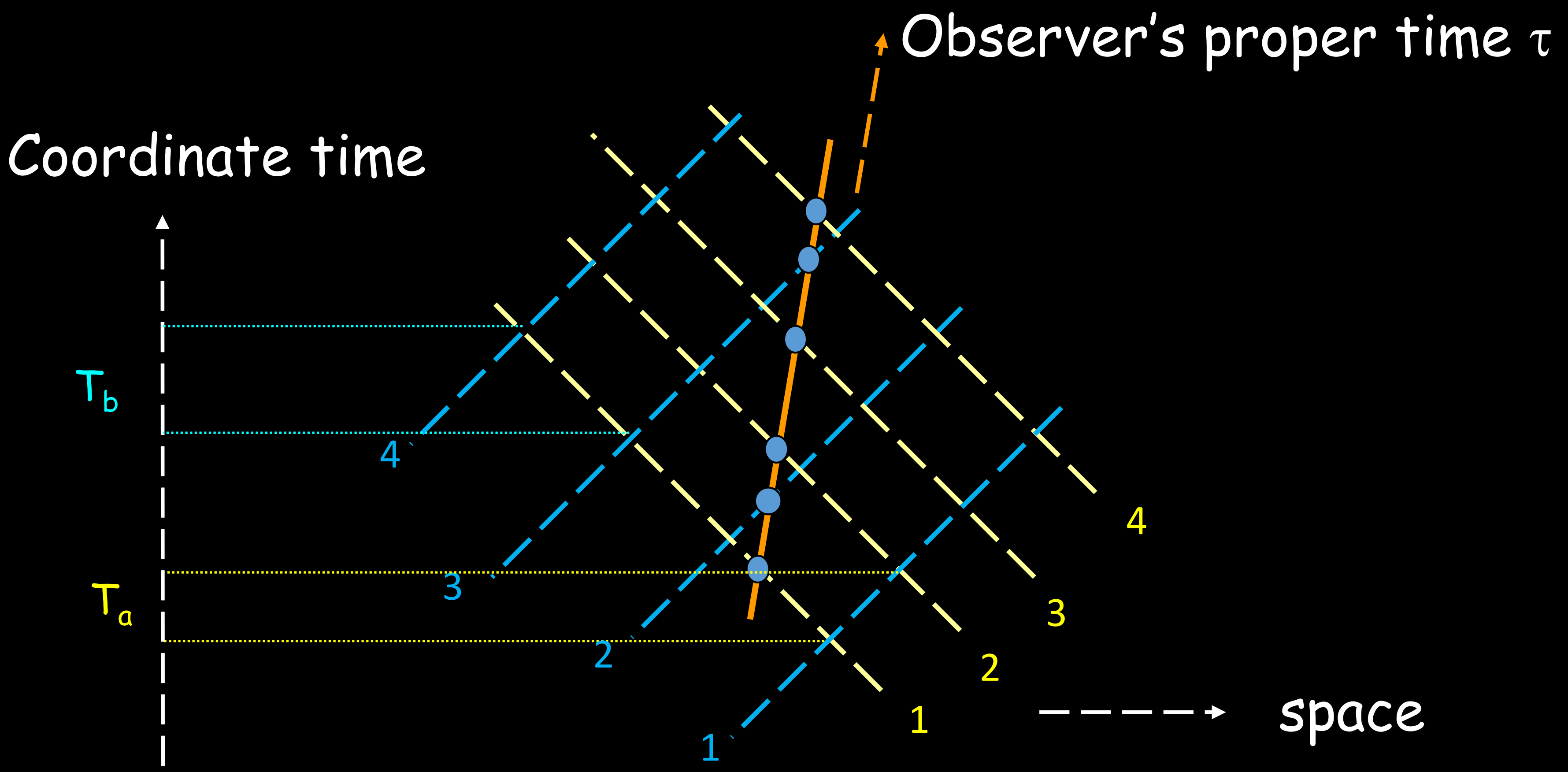
Relativistic positioning and geometry



Null geodesics are a faithfull texture for a curved space time and may be used to localized an event, once a reference frame has been chosen.

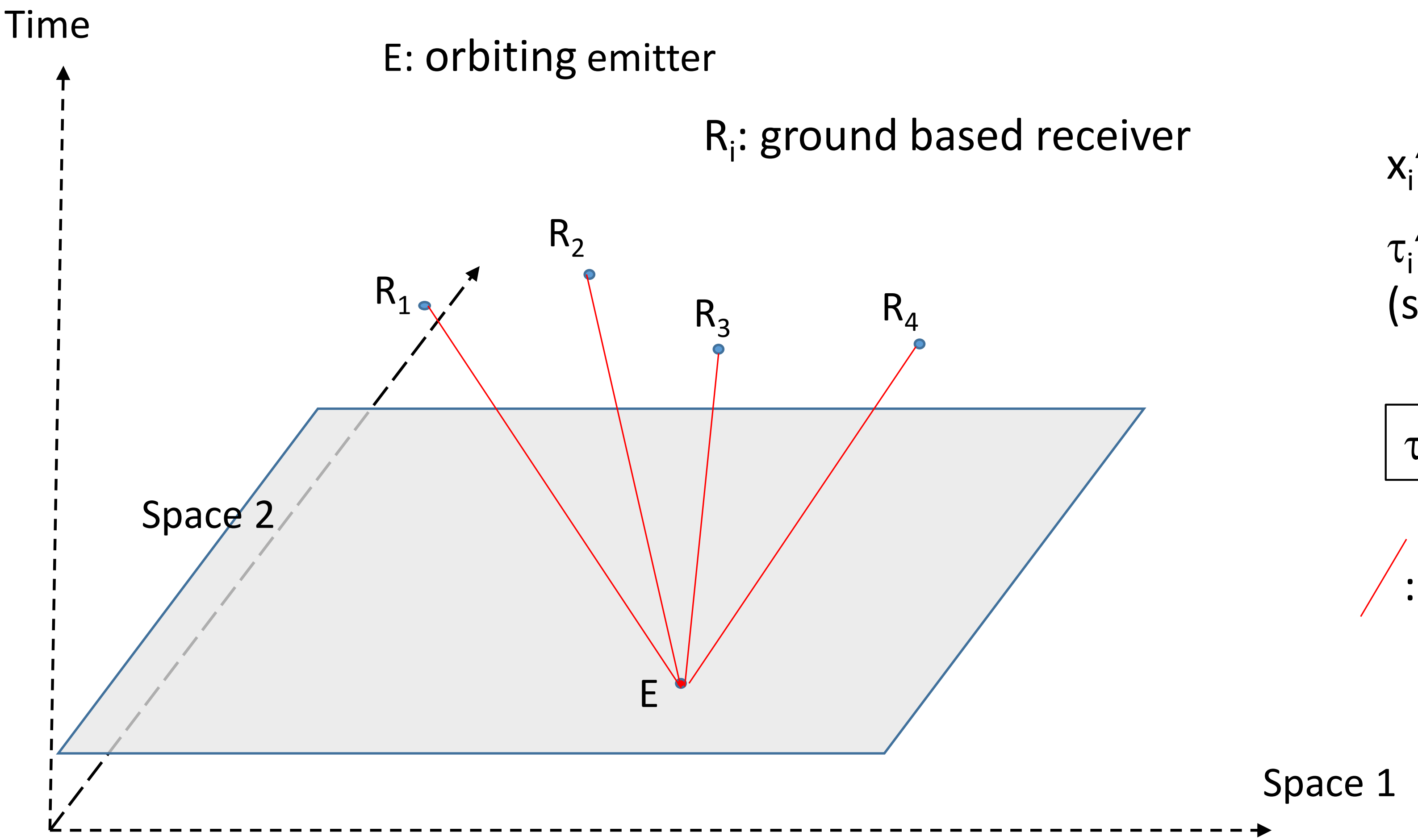
Self-positioning for navigation



Arrival times sequence measured on board

A. Tartaglia, ML. Ruggiero, E. Capolongo, *A null frame for spacetime positioning by means of pulsating sources*, Advances in Space Research, **47**, 645-653 (2011)

Bottom up (Minkowski)



x_i's: known coordinates

τ_i's: measured proper times (synchronous clocks)

τ_E, x_E's: to be determined

: light rays

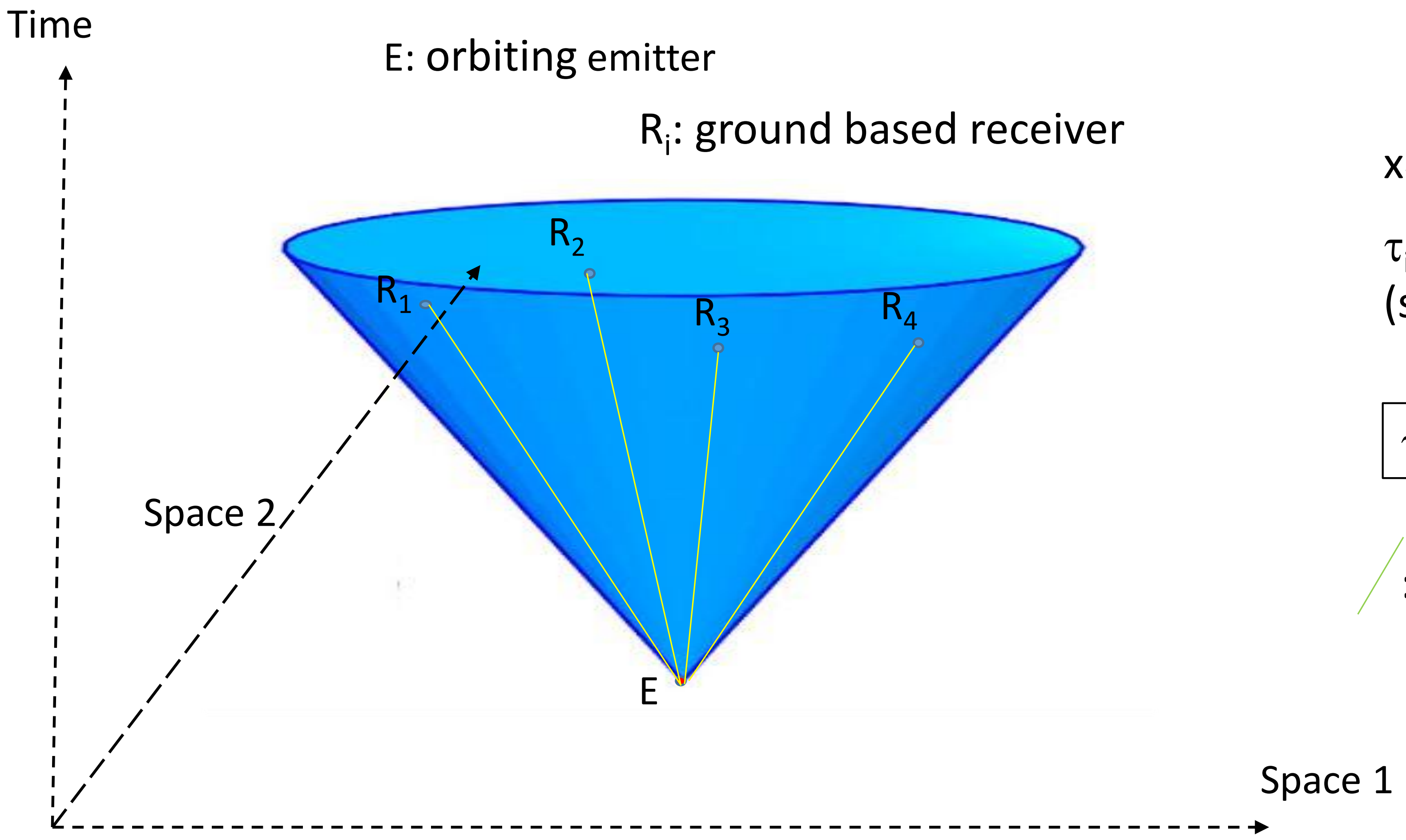
Matteo Luca Ruggiero, Angelo Tartaglia, Lorenzo Casalino, *Geometric definition of emission coordinates*, *Advances in Space Research*, **69**(12), 4221-4227

Future light cone of the emitter

G4S Galileo for Science

E: orbiting emitter

R_i: ground based receiver



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Basic equations

Null geodesic between the emitter and the i_{th} receiver (Cartesian coordinates, Earth centered non rotating frame):

$$c^2(t_i - t_E)^2 = (x_i - x_E)^2 + (y_i - y_E)^2 + (z_i - z_E)^2 \qquad i : 1 \div 5$$

x_i's, y_i's, z_i's are given t_i's are measured t_E, x_E, y_E, z_E are to be calculated

Subtracting the first from the other equations ends up with 4 linear eq.s for the four unknowns:

$$2c^2 t_E(t_j - t_1) - 2x_E(x_j - x_1) - 2y_E(y_j - y_1) - 2z_E(z_j - z_1) = c^2(t_1^{\,2} - t_j^{\,2}) + r_j^{\,2} - r_1^{\,2}$$

$$j : 2 \div 5$$

Analytical solution

$$\xi:\begin{pmatrix} t_E \\ x_E \\ y_E \\ z_E \end{pmatrix} \qquad C:2\begin{pmatrix} c^2(t_2-t_1) & x_1-x_2 & y_1-y_2 & z_1-z_2 \\ c^2(t_3-t_1) & x_1-x_3 & y_1-y_3 & z_1-z_3 \\ c^2(t_4-t_1) & x_1-x_4 & y_1-y_4 & z_1-z_4 \\ c^2(t_5-t_1) & x_1-x_5 & y_1-y_5 & z_1-z_5 \end{pmatrix} \qquad N:\begin{pmatrix} c^2\left(t_1{}^2-t_2{}^2\right)+r_2{}^2-r_1{}^2 \\ c^2\left(t_1{}^2-t_3{}^2\right)+r_3{}^2-r_1{}^2 \\ c^2\left(t_1{}^2-t_4{}^2\right)+r_4{}^2-r_1{}^2 \\ c^2\left(t_1{}^2-t_5{}^2\right)+r_5{}^2-r_1{}^2 \end{pmatrix}$$

$$C\cdot\xi = N \quad \Rightarrow \quad \xi = C^{-1}\cdot N$$

Include Earth's rotation

$$\begin{cases} x_1 - x_j = r_1 \sin\vartheta_1 \cos(\Omega t_1) - r_j \sin\vartheta_j \cos(\Omega t_j) \\ y_1 - y_j = r_1 \sin\vartheta_1 \sin(\Omega t_1) - r_j \sin\vartheta_j \sin(\Omega t_j) \\ z_1 - z_j = r_1 \cos\vartheta_1 - r_j \cos\vartheta_j \end{cases}$$

$$r_j{}^2 = x_j{}^2 + y_j{}^2 + z_j{}^2$$

Gravitational effects

Schwarzschild approximated symmetry

Radial null trajectory

$$m = \frac{GM_{Earth}}{c^2}$$

$$c(t_1 - t_E) = r_E - r_1 + 2m \ln \frac{r_E - 2m}{r_1 - 2m}$$

$$\frac{\Delta(t_i - t_E)}{t_i - t_E} = 2\frac{m}{r_E - r_i} \ln \frac{r_E - 2m}{r_1 - 2m} \cong 2\frac{m}{r_E - r_i} \ln \frac{r_E}{r_i} + O\left(\frac{m^2}{r^2}\right) \approx 6 \times 10^{-10}$$

Rotation of receivers and proper times

$$t_i = \frac{\tau_i}{\sqrt{1 - 2\frac{m}{r_i} - \frac{\Omega^2 r^2}{c^2}\sin^2\theta}} \approx \left[1 + \frac{m}{r_i} + \frac{\Omega^2 r_i^2}{2c^2}\sin^2\theta_i + \left(\frac{m^2}{r^2{}_{earth}}, \frac{\Omega^3 r^3{}_{earth}}{c^3}\right)\right]\tau_i$$

Uncertainties and errors

Position of the receivers on the ground → Systematic errors: Δx_i,…

Arrival times measurement → Measurement uncertainty: $\delta\tau_i$

Then, of course, one has the propagation through the atmosphere

Numerical implementation is under way

Thank you